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Theoretical and numerical analysis of problems with an interior turning point and a variable diffusion coefficient

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The paper discusses a two-point boundary-value problem with an interior turning point and a quadratic diffusion coefficient. After establishing bounds on solution derivatives, layer-eliminating coordinate transformations and the corresponding layer-resolving grids are constructed. The problem is discretized on such grids using the upwind scheme. The convergence of the numerical solution is analyzed.
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Introduction

The following boundary-value problem for an equation with a second order with a small parameter ε is tipical for the theoretical study of qualitative features arising in solutions to problems with layers along a coordinate x transversal to the layers:

$$-(\varepsilon + d(x))^{\nu} u'' + a(x)u' + c(x)u = f(x), \quad l_0 < x < l_1, \quad u(l_0, \varepsilon) = A_0, \quad u(l_1, \varepsilon) = A_1, \quad (1)$$

where $1 \gg \varepsilon > 0$, $d(x) \geq 0$, $\nu > 0$. Such a model problem allows one to get some idea of the issues associated with real physical processes, in particular, those modelled by the Navier – Stokes equations.

The case with a constant diffusion coefficient ($d(x) = 0$) is widely studied in the literature [1–5]. A problem of this type with $d(x) = x$, $\nu = 1$ was formulated in the monograph by Polubarinova – Kochina [6] to model filtration of a liquid in the neighbourhood of a circular orifice of small radius $r = \varepsilon$, while that with $\nu = 2$, $d(x) = x$ appears in the physics of motion of charges viewed as classical particles [7]. This problem for $d(x) = x$, $\nu = 1$, $l_0 = 0$ and arbitrary $a(x)$, while for $\nu \geq 2$ but without a turning point, i. e., when $a(0) > 0$, was analysed theoretically in [8, sect. 3.4]. An evolutionary problem related to (1), with $d(x) = x$, $\nu = 2$, was originally investigated numerically in [9] by using special grids.

To date, the following types of layers have been discovered to solve the problem (1): exponential, power of types 1 and 2, logarithmic, and hybrid-type layers (see [8, sect. 1.4, 1.5] and [10, sect. 2.1]). Of course, solutions to problem (1), not to mention to the Navier – Stokes equations, may have new types of layers that have not yet been discovered. The most popular are exponential layers. However, because they are the most narrow among the layers discovered, layer-resolving grids contrived for solving problems having exponential layers are not suitable for solving problems having non-exponential layers.

We demonstrate in the current paper that solutions to the problem (1) with $\nu = 1$, $d(x) = x^2$, $l_0 = -1$, $l_1 = 1$, and $a(0) = 0$ exhibit either an interior power-of-type-2 layer or a hybrid interior layer which is a combination of power-of-first-type and power-of-second-type layers (see [10, sect. 2.1]), depending on $c(0)$ and $a'(0)$. We construct layer-eliminating coordinate transformations $x(\xi, \varepsilon)$ and corresponding layer-resolving grids $x_i = x(i/N, \varepsilon)$, and analyze the convergence of numerical solutions obtained by the upwind scheme on the layer-resolving grids.

1. Estimates of derivatives

We assume in (1) $\nu = 1$, $d(x) = x^2$, $l_0 = -1$, $l_1 = 1$, i. e.,

$$\begin{gathered} L[u] \equiv -(\varepsilon + x^2)u'' + a(x)u' + c(x)u = f(x), \quad -1 \le x \le 1, \\ \Gamma[u] \equiv (u(-1, \varepsilon), u(1, \varepsilon)) = (A_0, A_1), \end{gathered} \tag{2}$$

where $1 \gg \varepsilon > 0$, $a(0) = 0$, $a(x),\ c(x),\ f(x) \in C^n[0, 1]$, $c(x) > 0$, $-1 \le x \le 1$.

1.1. Preliminary estimates of derivatives

It is well known that the pair (L, Γ) in (2) is inverse-monotone, i. e., if for two functions $u(x, \varepsilon)$ and $v(x, \varepsilon)$, $-1 \le x \le 1$,

$$(L, \Gamma)[u] \le (L, \Gamma)[v], \quad -1 \le x \le 1, \quad \text{then} \quad u(x, \varepsilon) \le v(x, \varepsilon), \quad -1 \le x \le 1.$$

This results in ε-uniform bounds on a solution $u(x, \varepsilon)$ to (2):

$$|u(x, \varepsilon)| \le M, \quad -1 \le x \le 1. \tag{3}$$

In this equation and hereafter, by m, M, m_i, M_j we designate positive constants independent of ε.

If $u(x, \varepsilon)$ is a solution to (2), then for $x_0 \ge 0$, $x \ge 0$,

$$u'(x, \varepsilon) - u'(x_0, \varepsilon) = \int\limits_{x_0}^{x} \frac{a(\xi)u'(\xi, \varepsilon)}{\varepsilon + \xi^2} \mathrm{d}\xi + \int\limits_{x_0}^{x} \frac{c(\xi)u(\xi, \varepsilon) - f(\xi)}{\varepsilon + \xi^2} \mathrm{d}\xi. \tag{4}$$

For the first integral in (4) we have

$$\int\limits_{x_0}^{x} \frac{a(\xi)u'(\xi, \varepsilon)}{\varepsilon + \xi^2} \mathrm{d}\xi = \frac{a(\xi)u(\xi, \varepsilon)}{\varepsilon + \xi^2}\bigg|_{x_0}^{x} - \int\limits_{x_0}^{x} \left(\frac{a(\xi)}{\varepsilon + \xi^2}\right)' u(\xi, \varepsilon) \mathrm{d}\xi. \tag{5}$$

Further, we will use the following obvious estimate:

$$(\varepsilon^{1/2} + |x|) \le \sqrt{2}(\varepsilon + x^2)^{1/2} \le \sqrt{2}(\varepsilon^{1/2} + |x|), \quad -1 \le x \le 1. \tag{6}$$

As $a(0)=0$, so

$$\left.\frac{a(\xi)u(\xi,\varepsilon)}{\varepsilon+\xi^2}\right|_{x_0}^{x} \le \frac{M}{\varepsilon^{1/2}+x}+\frac{M}{\varepsilon^{1/2}+x_0}, \quad x_0\ge 0, \quad x\ge 0,$$
$$\left|\left(\frac{a(\xi)}{\varepsilon+\xi^2}\right)'\right| = \left|\frac{a'(\xi)(\varepsilon+\xi^2)-2\xi a(\xi)}{(\varepsilon+\xi^2)^2}\right| \le \frac{M_1}{\varepsilon+\xi^2} \le \frac{M_1}{(\varepsilon^{1/2}+\xi)^2}, \quad \xi\ge 0, \tag{7}$$

and therefore, from (3), (5)–(7) we get

$$\left|\int\limits_{x_0}^{x} \frac{a(\xi)u'(\xi,\varepsilon)}{\varepsilon+\xi^2}\mathrm{d}\xi\right| \le \frac{M_2}{\varepsilon^{1/2}+x}+\frac{M_3}{\varepsilon^{1/2}+x_0}, \quad x_0\ge 0, \quad x\ge 0. \tag{8}$$

Similarly, from (3) and (6) it is obvious that

$$\left|\int\limits_{x_0}^{x} \frac{c(\xi)u(\xi,\varepsilon)-f(\xi)}{\varepsilon+\xi^2}\mathrm{d}\xi\right| \le \frac{M_4}{\varepsilon^{1/2}+x}+\frac{M_5}{\varepsilon^{1/2}+x_0}, \quad x_0\ge 0, \quad x\ge 0. \tag{9}$$

Taking $x_0\ge 1/2$ and satisfying, in accordance with (3), $u'(x_0,\varepsilon)\le M$, we get from (4), (8), and (9) $|u'(x,\varepsilon)| \le \dfrac{M}{\varepsilon^{1/2}+x}$, $0\le x\le 1$, and from (2), (3) and (6), taking into account $a(0)=0$, we readily obtain that

$$|u^{(i)}(x,\varepsilon)| \le \frac{M}{(\varepsilon^{1/2}+x)^i}, \quad n+1\ge i\ge 0, \quad 0\le x\le 1, \tag{10}$$

for some $M>0$. Similarly, for $-1\le x\le 0$ we get

$$|u^{(i)}(x,\varepsilon)| \le \frac{M}{(\varepsilon^{1/2}-x)^i}, \quad n+1\ge i\ge 0, \quad -1\le x\le 0,$$

and, using (6), the global estimate

$$|u^{(i)}(x,\varepsilon)| \le \frac{M}{(\varepsilon^{1/2}+|x|)^i} \le \frac{M}{(\varepsilon+x^2)^{i/2}}, \quad n+1\ge i\ge 0, \quad -1\le x\le 1. \tag{11}$$

As quantity $\int\limits_0^1 \dfrac{1}{\varepsilon^{1/2}+x}\mathrm{d}x = \ln(\varepsilon^{1/2}+1)-\ln\varepsilon^{1/2}$ is not uniformly bounded, neither estimate (10) nor (11) is very good, since, in accordance with formula (2.26) from [8] for the first derivative of a solution to (2), the following inequality is true: $\int\limits_0^1 |u'(x,\varepsilon)|\mathrm{d}x \le M$, i. e., the variation of the solution $u(x,\varepsilon)$ on the interval $[0,1]$ is uniformly bounded. Therefore, for the purpose of defining layer-damping transformations $x(\xi,\varepsilon):[-1,1]\to[-1,1]$ applied for generating layer-resolving grids by the formula $x_i=x(i/N,\varepsilon)$, $i=-N,-N+1,\dots,0,1,\dots,N$, we must improve estimate (10), and estimate (11) in consequence, so that

$$|u'(x,\varepsilon)|\le\phi(x,\varepsilon) \quad \text{and} \quad \int\limits_{-1}^{1}\phi(x,\varepsilon)\mathrm{d}x\le M. \tag{12}$$

We introduce further a designation a for $a'(0)$.

1.2. Case $c(0) + a > 0$

Estimate (11) is easily improved in the case $c(0) + a > 0$ since, in this case, $c(x) + a'(x) > 0$, $|x| \le m_0$, for some $m_0 > 0$, therefore, the pair (L_1, Γ), where

$$L_1[v](x,\varepsilon) = -(\varepsilon + x^2)v'' + a_1(x)v' + c_1(x)v, \quad a_1(x) = a(x) - 2x, \quad c_1(x) = c(x) + a'(x), \tag{13}$$

is inverse monotone on the interval $|x| \le m_0$. For $v(x,\varepsilon) = u'(x,\varepsilon)$, where $u(x,\varepsilon)$ is a solution of (2), we have

$$|L_1[u'](x,\varepsilon)| = |f'(x) - c'(x)u(x,\varepsilon)| \le M, \quad |x| \le m_0.$$

Since, in accordance with (11), $|u'(-m_0), \varepsilon)| \le M$ and $|u'(m_0), \varepsilon)| \le M$, taking a sufficiently large positive constant M as a barrier function for the pair (L_1, Γ) yields $|u'(x,\varepsilon)| \le M$, $|x| \le m_0$, thus, from (11) we conclude

$$|u'(x,\varepsilon)| \le M, \quad -1 \le x \le 1.$$

Further, similarly to proof of (11), we come, using equation

$$L_1[u'](x,\varepsilon) = f'(x) - c'(x)u(x,\varepsilon), \quad -1 \le x \le 1,$$

to the estimate:

$$|u^{(i)}(x,\varepsilon)| \le \frac{M}{(\varepsilon^{1/2} + |x|)^{i-1}} \le \frac{M}{(\varepsilon + x^2)^{(i-1)/2}}, \quad n+1 \ge i \ge 1, \quad -1 \le x \le 1, \tag{14}$$

when $a + c(0) > 0$. This estimate for $i = 1$ is subject to (12).

The same technique, applied sequentially step by step, can be used to prove the following theorem.

Theorem 1.1. *Let $u(x,\varepsilon)$ be a solution to (2) with the following condition:*

$$c(0) + ia - 2(i-1) > 0, \quad \text{for some } k\text{: } 1 \le k \le n \text{ and all } i\text{: } 1 \le i \le k \le n, \tag{15}$$

then,

$$|u^{(i)}(x,\varepsilon)| \le M[1 + (\varepsilon^{1/2} + |x|)^{k-i}] \le M[1 + (\varepsilon + x^2)^{(k-i)/2}], \quad 1 \le i \le n, \quad -1 \le x \le 1. \tag{16}$$

Thus, in the case of (15), solution derivatives of (2) up to k are uniformly bounded.

For proving estimate (16), at the ith step we use a generalization of operator (13):

$$L_i[v](x,\varepsilon) = -(\varepsilon + x^2)v'' + a_i(x)v' + c_i(x)v, \quad i \ge 1,$$

where $a_i(x) = a(x) - 2ix$, $c_i(x) = c(x) + ia'(x) - 2(i-1)$. For this operator we have $|L_i[u^{(i)}](x,\varepsilon)| \le M$ if (15) holds, and consequently $|u^{(j)}(x,\varepsilon)| \le M$, $1 \le j \le i$, $-1 \le x \le 1$. Similarly to proof of (14), we come to estimate (16).

1.3. Estimates of $c(x)u(x,\varepsilon)-f(x)$ for $c(0)+a\le 0$

It appears that for obtaining an estimate of $u^{(i)}(x,\varepsilon)$ (more accurate than (11)) in the case $c(0)+a\le 0$, we need to find necessary bounds for the function $c(x)u(x,\varepsilon)-f(x)$. It is evident that $|c(x)u(x,\varepsilon)-f(x)|\le M$ because of (3). In order to improve this estimate, when $u(x,\varepsilon)$ is a solution to (2) with the condition $a(0)=0$, we use preliminary estimate (11) for $i=1$, and the pair (L,Γ) from (2). If $u(x,\varepsilon)$ is a solution to (2), then

$$L[cu-f](x,\varepsilon)=-(\varepsilon+x^2)[2c'(x)u'(x,\varepsilon)+c''u(x,\varepsilon)-f''(x)]+a(x)[c'(x)u(x,\varepsilon)-f'(x)], \quad (17)$$

and taking into account (3) and (11) for $i=1$, and the condition $a(0)=0$, we find from (17) and (7)

$$|L[cu-f](x,\varepsilon)|\le M(\varepsilon^{1/2}+|x|)\le M(\varepsilon+x^2)^{1/2},\quad -1\le x\le 1, \quad (18)$$

for some $M>0$. Now, for estimating $c(x)u(x,\varepsilon)-f(x)$ we introduce the barrier function

$$b(x,\varepsilon)=(\varepsilon+x^2)^\beta,\quad 1/2\ge\beta>0,\quad -1\le x\le 1.$$

We have

$$b'(x,\varepsilon)=2\beta x(\varepsilon+x^2)^{\beta-1},\quad b''(x,\varepsilon)=2\beta(\varepsilon+x^2)^{\beta-1}+4\beta(\beta-1)x^2(\varepsilon+x^2)^{\beta-2}.$$

Therefore,

$$\begin{aligned} L[b](x,\varepsilon)=&\left(c(x)-2\beta+2\beta(a+2(1-\beta))\frac{x^2}{\varepsilon+x^2}\right)(\varepsilon+x^2)^\beta+\\ &+\frac{2\beta(xa(x)-ax^2)}{\varepsilon+x^2}(\varepsilon+x^2)^\beta,\quad -1\le x\le 1. \end{aligned} \quad (19)$$

As $|xa(x)-ax^2|\le Mx^3$, we conclude from (19) that if $c(0)>2\beta$ for $a\ge 0$ and $c(0)+2\beta(a-1)>0$ for $a<0$, then

$$|L[b](x,\varepsilon)|\ge M_0(\varepsilon+x^2)^\beta,\quad |x|\le m_0,$$

for some $M_0>0$ and $m_0>0$. Thus, taking into account (18) and the condition $0<\beta\le 1/2$, we have for the pair (L,Γ) from (2)

$$(L,\Gamma)[Mb](x,\varepsilon)\ge(L,\Gamma)[cu-f](x,\varepsilon)\ge(L,\Gamma)[-Mb](x,\varepsilon),\quad -m_0\le x\le m_0,$$

for some large $M>0$, which results in

$$|c(x)u(x,\varepsilon)-f(x)|\le M(\varepsilon+x^2)^\beta,\quad -m_0\le x\le m_0.$$

Taking into account (3) we can conclude that

$$|c(x)u(x,\varepsilon)-f(x)|\le M(\varepsilon+x^2)^\beta,\quad -1\le x\le 1, \quad (20)$$

where β is a positive number satisfying $0<\beta\le 1/2$ and $c(0)-2\beta>0$ if $a\ge 0$; $0<\beta\le 1/2$ and $c(0)+2\beta(a-1)>0$ if $a<0$.

1.4. Estimates of first and higher derivatives for $c(0) + a \le 0$

By resolving (2) with respect to $u'(x,\varepsilon)$ we obtain

$$u'(x,\varepsilon) = u'(x_0,\varepsilon)\exp[\psi(x_0,x,\varepsilon)] + \int_{x_0}^{x} \frac{c(\xi)u(\xi,\varepsilon) - f(\xi)}{\varepsilon + \xi^2}\exp[\psi(\xi,x,\varepsilon)]\mathrm{d}\xi, \quad -1 \le x \le 1, \tag{21}$$

where

$$\psi(\xi,x,\varepsilon) = \int_{\xi}^{x} \frac{a(\eta)}{\varepsilon + \eta^2}\mathrm{d}\eta.$$

Case $c(0) + a$. In this case $a < 0$. We have for $\psi(\xi,x,\varepsilon)$ in (21)

$$\psi(\xi,x,\varepsilon) = \int_{\xi}^{x} \left(\frac{a\eta}{\varepsilon + \eta^2} + \frac{a(\eta) - a\eta}{\varepsilon + \eta^2}\right)\mathrm{d}\eta = \frac{a}{2}\ln\left(\frac{\varepsilon + x^2}{\varepsilon + \xi^2}\right) + g_1(\xi,x,\varepsilon), \tag{22}$$

where

$$g_1(\xi,x,\varepsilon) = \int_{\xi}^{x} \frac{a(\eta) - a\eta}{\varepsilon + \eta^2}\mathrm{d}\eta.$$

As $|a(\eta) - a\eta| \le M\eta^2$, so $|g_1(\xi,x,\varepsilon)| \le M$, $-1 \le \xi,\ x \le 1$, therefore, using (22) we obtain

$$\exp[\psi(\xi,x,\varepsilon)] \le M\left(\frac{\varepsilon + x^2}{\varepsilon + \xi^2}\right)^{a/2}, \quad -1 \le \xi,\ x \le 1. \tag{23}$$

In accordance with (20)

$$|c(x)u(x,\varepsilon) - f(x)| \le M(\varepsilon + x^2)^{\beta}, \quad -1 \le x \le 1,$$

for β satisfying both $0 < \beta \le 1/2$ and $c(0) + 2\beta(a-1) > 0$, as $a < 0$, therefore, we get, using also (23) and (6) for $0 \le x_0,\ \xi,\ x \le 1$

$$\begin{aligned}
&\int_{x_0}^{x} \frac{c(\xi)u(\xi,\varepsilon) - f(\xi)}{\varepsilon + \xi^2}\exp[g_1(\xi,x,\varepsilon)]\mathrm{d}\xi \le M\int_{x_0}^{x} \frac{(\varepsilon + \xi^2)^{\beta}}{\varepsilon + \xi^2}\left(\frac{\varepsilon + x^2}{\varepsilon + \xi^2}\right)^{a/2}\mathrm{d}\xi = \\
&= M(\varepsilon + x^2)^{a/2}\int_{x_0}^{x} (\varepsilon + \xi^2)^{\beta - a/2 - 1}\mathrm{d}\xi \le M_1(\varepsilon + x^2)^{a/2}\int_{x_0}^{x} (\varepsilon^{1/2} + \xi)^{2\beta - a - 2}\mathrm{d}\xi \le \\
&\le M_2(\varepsilon + x^2)^{a/2}((\varepsilon^{1/2} + x)^{2\beta - a - 1} + (\varepsilon^{1/2} + x_0)^{2\beta - a - 1}) \le \\
&\le M_3\left((\varepsilon + x^2)^{\beta - 1/2} + \frac{(\varepsilon + x_0^2)^{\beta - a/2 - 1/2}}{(\varepsilon + x^2)^{-a/2}}\right),
\end{aligned} \tag{24}$$

with an additional $2\beta - a \ne 1$ restriction on β. Thus, from (21), (23), and (24) we get for $c(0) + a \le 0$, β satisfying both $0 < \beta \le 1/2$, $2\beta - a \ne 1$, and $c(0) + 2\beta(a-1) > 0$:

$$|u'(x,\varepsilon)| \le M\left(u'(x_0,\varepsilon)\left(\frac{\varepsilon + x^2}{\varepsilon + x_0^2}\right)^{a/2} + (\varepsilon + x^2)^{\beta - 1/2} + \frac{(\varepsilon + x_0^2)^{\beta - a/2 - 1/2}}{(\varepsilon + x^2)^{-a/2}}\right), \quad 0 \le x,\ x_0 \le 1. \tag{25}$$

Case $1 > -a > 0$, $c(0) + a \le 0$. In this case, assuming $x_0 > 1/2$ in (25), which results in $|u'(x_0, \varepsilon)| \le M$, we get, using (6),

$$|u'(x,\varepsilon)| \le M((\varepsilon+x^2)^{a/2}+(\varepsilon+x^2)^{\beta-1/2}) \le M((\varepsilon^{1/2}+x)^a+(\varepsilon^{1/2}+x)^{2\beta-1}), \quad 0 \le x \le 1, \tag{26}$$

where β satisfies both $0 < \beta \le 1/2$ and $2\beta - a \ne 1$, and $c(0) + 2\beta(a-1) > 0$.

Case $-a > 1$, $c(0) + a \le 0$. In this case, assuming $x_0 = 0$, we get from (25), taking into account estimate $|u'(0,\varepsilon)| \le M\varepsilon^{-1/2}$ from (10),

$$\begin{aligned} |u'(x,\varepsilon)| &\le M\left(\frac{\varepsilon^{-(a+1)/2}}{(\varepsilon+x^2)^{-a/2}} + (\varepsilon+x^2)^{\beta-1/2} + \frac{\varepsilon^{\beta-(a+1)/2}}{(\varepsilon+x^2)^{-a/2}}\right) \le \\ &\le M\left(\frac{\varepsilon^{-(a+1)/2}}{(\varepsilon+x^2)^{-a/2}} + (\varepsilon+x^2)^{\beta-1/2}\right) \le \\ &\le M\left(\frac{\varepsilon^{-(a+1)/2}}{(\varepsilon^{1/2}+x)^{-a}} + (\varepsilon^{1/2}+x)^{2\beta-1}\right), \quad 0 \le x \le 1. \end{aligned} \tag{27}$$

Estimates similar to (26) and (27) are, in the same manner, easily proved for $-1 \le x \le 0$. Therefore, we obtain the following global estimate of the first derivative for $-1 \le x \le 1$, taking into account (6)

$$|u'(x,\varepsilon)| \le M \begin{cases} (\varepsilon^{1/2}+|x|)^{\beta_1-1} + (\varepsilon^{1/2}+|x|)^{2\beta-1}, & 0 < -a < 1, \\ \dfrac{\varepsilon^{\alpha_0/2}}{(\varepsilon^{1/2}+|x|)^{\alpha_0+1}} + (\varepsilon^{1/2}+|x|)^{2\beta-1}, & 1 < -a, \end{cases} \tag{28}$$

where $\beta_1 = a+1$, $\alpha_0 = -(a+1)$, β satisfies both $0 < \beta \le 1/2$, $2\beta - a \ne 1$, and $c(0) + 2\beta(a-1) > 0$.

Further, using (2) and estimates (20) and (28), we readily come to

Theorem 1.2. *Let $u(x,\varepsilon)$ be a solution of (2) with the condition $c(0) + a \le 0$, then for $-1 \le x \le 1$*

$$|u^{(i)}(x,\varepsilon)| \le M \begin{cases} (\varepsilon^{1/2}+|x|)^{\gamma-i}, & 0 < -a < 1, \\ \dfrac{\varepsilon^{\alpha_0/2}}{(\varepsilon^{1/2}+|x|)^{\alpha_0+i}} + (\varepsilon^{1/2}+|x|)^{2\beta-i}, & -a > 1, \end{cases} \tag{29}$$

where $\gamma = \min\{a+1, 2\beta\}$, $\alpha_0 = -(a+1)$, and β is an arbitrary positive number satisfying both $0 < \beta \le 1/2$, $2\beta - a \ne 1$, and $c(0) + 2\beta(a-1) > 0$.

Thus, solutions to problem (2) may have either power-of-type-2 interior layers or hybrid ones which are combinations of power-of-type-1 and power-of-type-2 layers.

The proofs of Theorems 1.1 and 1.2 may serve as guidance for establishing estimates of solution derivatives when the diffusion coefficient is of the form $\varepsilon + x^{2i}$, $i > 1$, and in the case of a boundary turning point, i. e., when $0 \le x \le 1$ in (2).

1.5. Transformations eliminating hybrid layers

The numerical algorithm advocated in this paper for solving (2) is based on piece-wise smooth layer-damping coordinate transformations $x(\xi,\varepsilon) : [-1,1] \to [-1,1]$ in accordance with a basic principle: they are to eliminate singularities of high order of solutions $u(x,\varepsilon)$ at each interval $[a_j, b_j]$ of smoothness; i. e., the high-order derivatives of any particular solution with respect to the new coordinate ξ are to have the following bounds:

$$\left|\frac{\mathrm{d}^i}{\mathrm{d}\xi^i}u[x(\xi,\varepsilon),\varepsilon]\right| \le M, \quad i \le n,\ a_j \le \xi \le b_j,$$

where the constant M does not depend on the parameter ε, and the number n depends on the order of the approximation of the problem: the higher the order, the larger the number n will be. With the help of such transformations, any problem can be solved using high-order approximations in the physical interval x on layer-resolving grids defined by mapping the nodes of a uniform grid with suitable coordinate transformations $x(\xi,\varepsilon)$, as in [10, 11].

To eliminate a hybrid singularity

$$\varepsilon^{k\alpha}/(\varepsilon^k + x)^{\alpha+i} + (\varepsilon^k + x)^{b-i}, \quad 0 \le x \le 1, \tag{30}$$

up to order n, which combines power-of-first- and power-of-second-type singularities, such as the one in (29) for $k = 1/2$ and $-a > 1$, we use a combination of two coordinate transformations, one of which eliminates power-of-type-1 layers and the other eliminates power-of-type-2 layers. A transformation designated as $x_{p1}(\xi,\varepsilon,p,k)$, of the class $C^l[0,1]$, $l \le n$, was described in [10, 11] for eliminating power-of-type-1 singularities $\dfrac{\varepsilon^{k\alpha}}{(\varepsilon^k + x)^{\alpha+i}}$, $0 \le x \le 1$, up to order n, and has the following form:

$$x_{p1}(\xi,\varepsilon,p,k) = \begin{cases} c_1\varepsilon^k((1-d\xi)^{-1/p} - 1), & 0 \le \xi \le \xi_0^n, \\ c_1\left[\varepsilon^{k(1-v/p)} - \varepsilon^k + \left(\dfrac{\varepsilon^k}{(1-d\xi)^{1/p}}\right)'(\xi_0^n)(\xi-\xi_0^n) + \right. & \\ +\dfrac{1}{2}\left(\dfrac{\varepsilon^k}{(1-d\xi)^{1/p}}\right)''(\xi_0^n)(\xi-\xi_0^n)^2 + \ldots + & \\ \left. +\dfrac{1}{l!}\left(\dfrac{\varepsilon^k}{(1-d\xi)^{1/p}}\right)^{(l)}(\xi_0^n)(\xi-\xi_0^n)^l + c_0(\xi-\xi_0^n)^{l+1}\right], & \xi_0^n \le \xi \le 1, \end{cases} \tag{31}$$

where $d = (1-\varepsilon^{kv})/\xi_0^n$; $v = p/(1+np)$; p is an arbitrary positive constant satisfying $0 < p \le \alpha/n^2$; $0 < \xi_0^n < 1$ (for example $\xi_0^n = 1/2$); c_0 is an arbitrary positive constant; and $c_1 > 0$ is such that the necessary boundary condition $x_{p1}(1,\varepsilon,p,k) = 1$ is satisfied ($c_1 < 1/(c_0(1-\xi_0^n)^{l+1})$). For example, the transformation (31) with $k = 1$ eliminates the singularity $\varepsilon^\alpha/(\varepsilon + x)^{\alpha+i}$, $1 \le i \le n$. A simpler form of transformation (31) for $p = 1$ was published earlier in [12], and for an arbitrary $p > 0$ in [13]. Paper [14] shows that the grid obtained through transformation (31) by $x_i = x_{p1}(i/N,\varepsilon,p,1/2)$, is the most effective for numerical modelling of viscous flows over a plate, compared with results obtained on the grids more commonly used. Note that this transformation with an arbitrary $p > 0$ and $0 < v \le p/(1+np)$ eliminates the exponential singularity $(1/\varepsilon^{ik})\exp(-mx/\varepsilon^k)$ up to an arbitrary order n (see [10]).

A transformation designated as $x_{p2}(\xi,\varepsilon,t,k)$, for eliminating power-of-second-type singularities $(\varepsilon^k + x)^{b-i}$, $0 \le x \le 1$ up to order n, in particular, those in (29) for $k = 1/2$, $-a < 1$, was described in [10, 11], and has the following form:

$$x_{p2}(\xi,\varepsilon,t,k) = \frac{(\varepsilon^{kt} + \xi)^{1/t} - \varepsilon^k}{(\varepsilon^{kt} + 1)^{1/t} - \varepsilon^k}, \quad 0 \le \xi \le 1, \tag{32}$$

where $0 < t \leq \min\{b/n, 1/n\}$. Note that when $b \geq n$, the function $(\varepsilon + x)^{b-n}$ is ε-uniformly bounded, so that one can consider this singularity for $0 < b < n$ only. The numerical grid based on this transformation was employed in [15] for solving a singularly perturbed problem with an interior power-of-type-2 layer, to prove high-order uniform convergence in an integral norm using FEM.

Theorem 1.3. *Hybrid singularities (30) for $0 \leq x \leq 1$ are eliminated up to n by the coordinate transformation designated as $x_h(\xi, \varepsilon, \alpha, b, p, k)$:*

$$x_h(\xi, \varepsilon, \alpha, b, p, k) = \frac{(\varepsilon^{kt} + x_{p1}(\xi, \varepsilon, p, kt))^{1/t} - \varepsilon^k}{(\varepsilon^{kt} + 1)^{1/t} - \varepsilon^k}, \quad 0 < t \leq \min\{b/n, 1/n\}, \quad 0 < p \leq \alpha/(tn^2), \quad 0 \leq \xi \leq 1, \tag{33}$$

which is a composition of coordinate transformations (31) and (32), one eliminating power-of-type-1 layers and the other eliminating power-of-type-2 layers.

A proof of this theorem was given in [10, 11].

Note that $k = 1/2$ in (33) for estimates (16) and (29). Transformation (33) with $k = 1/2$ is suitable for eliminating not only hybrid singularity (29) for $-a > 1$, but also power-of-second-type singularities (16) and, in (29) for $0 < -a < 1$, assuming for these cases α in (33) as an arbitrary positive constant, so we can set $\alpha = 1$.

A layer-resolving grid to problem (2) with an interior turning point $x_0 = 0$ is defined by mapping a uniform grid with a coordinate transformation $x(\xi, \varepsilon, \alpha, b, p, 1/2) : [-1, 1] \to [-1, 1]$ based on (33):

$$x(\xi, \varepsilon, \alpha, b, p, 1/2) = \begin{cases} -x_h(-\xi, \varepsilon, \alpha, b, p, 1/2), & \xi \in [-1, 0], \\ x_h(\xi, \varepsilon, \alpha, b, p, 1/2), & \xi \in [0, 1]. \end{cases} \tag{34}$$

For a problem with an arbitrary interior turning point x_0 in the interval $[l_0, l_1]$, one can use an additional monotone function $\varphi_{x_0}(x)$, which maps the interval $[-1, 1]$ onto $[l_0, l_1]$ with the restrictions $\varphi_{x_0}(-1) = l_0$, $\varphi_{x_0}(0) = x_0$, $\varphi_{x_0}(1) = l_1$. The corresponding transformation for generating layer-resolving grids is defined as a composition of $x(\xi, \varepsilon, \alpha, b, p, 1/2)$ and $\varphi_{x_0}(x)$ [11, sect. 8.1.6].

2. Numerical algorithm and grids

2.1. Numerical algorithm

We use as an approximation of the singularly perturbed boundary-value problem (2) the standard upwind finite difference scheme on a nonuniform grid x_i, $i = -N, -N+1, \ldots, -1, 0, 1, \ldots, N$, $x_{-N} = -1 < x_{-N+1} < \ldots < x_N = 1$:

$$-\frac{2(\varepsilon+x_i)^2}{h_i+h_{i-1}} \left(\frac{u^h_{i+1}-u^N_i}{h_i} - \frac{u^h_i-u^N_{i-1}}{h_{i-1}} \right) + a_-(x_i)\frac{u^h_{i+1}-u^N_i}{h_i} + a_+(x_i)\frac{u^h_i-u^N_{i-1}}{h_{i-1}} + c(x_i)u_i = f(x_i), \quad i = -N+1, \ldots, -1, 0, 1, 2, \ldots, N-1, \quad u^N_{-N} = u_{-1}, \quad u^N_N = u_1, \tag{35}$$

where $h_i = x_{i+1} - x_i$ and $a_\pm(x) = (a(x) \pm |a(x)|)/2$. The nodes x_i, $i = -N, \ldots 0, \ldots, N$, of a layer-resolving grid are obtained explicitly using layer-damping transformation (34), namely,

$$x_i = x(ih, \varepsilon, \alpha, b, p, 1/2), \quad i = -N, \ldots, -1, 0, 1, \ldots, N, \quad h = 1/N.$$

Calculations of problem (2) are conducted for different values of ε: 10^{-6}, 10^{-8}. For each of these values, sequences of grids with doubled numbers of grid steps: $N_t = 2^t N_h$, $t = 0, 1, \dots$ are used, where N_h is the number for the rough grid. Usually $N_h = 160$, $t_{\max} = 6$. The numerical solution at the ith node of the grid related to N_t is designated by $u_i^{N_t}$, $i = 0, 1, \dots, N_t$.

For estimating the accuracy of the numerical algorithm, the following characteristic is introduced:

$$r_{t,\varepsilon} = \max_{-N_t \le i \le N_t} |u_i^{N_t} - u_{2i}^{N_{t+1}}|, \quad t = 0, 1, \dots .$$

In addition to this, one more characteristic,

$$du_{t,\varepsilon} = \max_{-N_t \le i \le N_t} |u_{i+1}^{N_t} - u_i^{N_t}|, \quad i = -N_t + 1, \dots, 0, 1, \dots, N_t - 1,$$

is introduced, which is related to the jump of the numerical solution at the neighbouring nodes.

The characteristic $r_{t,\varepsilon}$ is applied to estimate the order of accuracy of the numerical solution:

$$\beta_2 = \log_2(r_{t,\varepsilon}/r_{t+1,\varepsilon}), \quad t = 0, 1, \dots,$$

and, consequently, $du_{t,\varepsilon}$ to estimate the order of the numerical-solution jump in the neighbouring nodes

$$\beta_3 = \log_2(du_{t,\varepsilon}/du_{t+1,\varepsilon}), \quad t = 0, 1, \dots$$

Note that if a solution to (2) has neither boundary nor interior layers, then for the numerical solution of this problem the value β_2 is close to p_0, while β_3 is close to 1 through the use of a stable scheme of order p_0 on the uniform grid $x_i = ih$.

Theorem 2.1. *If $x_i = x(ih, \varepsilon, \alpha, b, p, 1/2)$, $i = -N, \dots, 0, \dots, N$, $h = 1/N$, where $x(\xi, \varepsilon, \alpha, b, p, 1/2)$ is defined by (34) for $l = n = 2$, then*

$$|u_i - u(x_i, \varepsilon)| \le M/N, \quad i = -N, \dots, N,$$

where M is independent of N.

The same result is proved in [8, subsect. 7.4.2], for problem (1) with $d(x) = 0$, $a(0) = 0$, $a'(0) < 0$, having hybrid layers.

2.2. Numerical experiments

This section presents results of numerical solutions to problem (2) obtained by scheme (35) on a grid $x_i = x(i/N, \varepsilon)$, $i = -N, -N+1, \dots, 0, 1, \dots, N$, where $x(\xi, \varepsilon) : [-1, 1] \to [-1, 1]$ is a coordinate transformation (34) for $l = n = 2$ in (31).

Example 1. For the first numerical experiment we consider the following problem:

$$-(\varepsilon + x^2) - 0.5xu' + u = \sin(3\pi x), \quad -1 \le x \le 1, \quad u(-1, \varepsilon) = -1, \quad u(1, \varepsilon) = 1.$$

For this problem $a = -0.5$, $c(0) = 1$, and so $\beta = 0.3$, $\nu = 0.5$ matches the requirements of Theorem 1.2 for $-a < 1$. Thus, estimate (29) is as follows:

$$|u^{(i)}(x, \varepsilon)| \le M(\varepsilon^{1/2} + |x|)^{0.5-i}, \quad -1 \le x \le 1.$$

In accordance with Theorem 2.1, the numerical grid obtained by coordinate transformation (34) with $t = 0.3$, $b = 0.5$, $p = 1$, $\alpha = 1$ provides uniform convergence by scheme (35). Table 1 and Fig. 1 show the values of characteristics β_2, β_3, and the numerical solution, for $\varepsilon = 10^{-6}$.

Table 1. Example 1

t	N	r	β_2	du	β_3
4	160	0.004250	1.260236	0.100860	0.926108
5	320	0.001748	1.281497	0.052253	0.948764
6	640	0.001010	0.791338	0.026653	0.971219
7	1280	0.000541	0.901324	0.013467	0.984854
8	2560	0.000280	0.951851	0.006770	0.992241
9	5120	0.000142	0.976335	0.003394	0.996074
10	10240	0.000072	0.988247	0.001699	0.998026

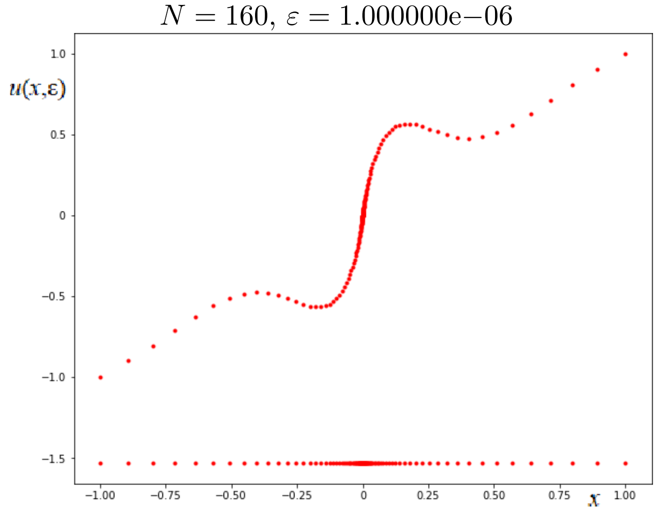

Fig. 1. Example 1

Table 2. Example 2

t	N	r	β_2	du	β_3
4	160	0.009469	0.891554	0.106472	0.965726
5	320	0.006065	0.642566	0.054238	0.973085
6	640	0.003414	0.829167	0.027482	0.980823
7	1280	0.001817	0.910086	0.013841	0.989549
8	2560	0.000937	0.955771	0.006947	0.994562
9	5120	0.000476	0.977440	0.003480	0.997228
10	10240	0.000240	0.988776	0.001742	0.998601

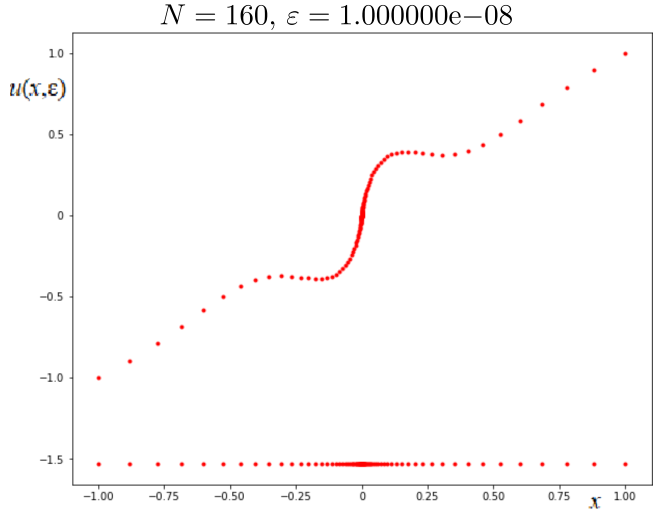

Fig. 2. Example 2

Example 2. For the second numerical experiment we consider the following problem:

$$-(\varepsilon + x^2) - 2xu' + 2u = \sin(3\pi x), \quad -1 \le x \le 1, \quad u(-1,\varepsilon) = -1, \quad u(1,\varepsilon) = 1.$$

For this problem $a = -2$, $c(0) = 2$, and so $\beta = 0.3$, $\alpha = 1$ match the requirements of Theorem 1.2 for $-a > 1$. Thus, estimate (29) is as follows:

$$|u^{(i)}(x,\varepsilon)| \le M \left(\frac{\varepsilon^{1/2}}{(\varepsilon^{1/2} + |x|)^{1+i}} + (\varepsilon^{1/2} + |x|)^{0.6-i} \right), \quad -1 \le x \le 1.$$

In accordance with Theorem 2.1, the numerical grid obtained by coordinate transformation (34) with $k = 0.5$, $t = 0.3$, $p = 10/12$, $b = 0.6$, $\alpha = 1$, provides uniform convergence by scheme (35). Table 2 and Fig. 2 show the values of characteristics β_2, β_3, and the numerical solution for, $\varepsilon = 10^{-8}$.

Conclusion

The paper establishes bounds on solution derivatives to a two-point boundary-value problem with an interior turning point and a quadratic diffusion coefficient. It describes construction of layer-eliminating coordinate transformations and the corresponding layer-resolving grids, and shows first-order uniform convergence of numerical solutions through an upwind scheme on the layer-resolving grids. Theoretical results are confirmed by numerical experiments.

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ВЫЧИСЛИТЕЛЬНЫЕ ТЕХНОЛОГИИ

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Теоретический и численный анализ задачи с внутренней точкой поворота и переменным коэффициентом диффузии

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Аннотация

Рассматривается двухточечная краевая задача с внутренней точкой поворота и квадратичным коэффициентом диффузии. Устанавливаются оценки производных решений, конструируются координатные преобразования, устраняющие слои и соответствующие им сгущающиеся в слоях разностные сетки. Анализируется сходимость численного решения для схемы с направленными разностями на полученной разностной сетке.

Ключевые слова: малый параметр, точка поворота, внутренний слой, сходимость.

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